3.2 Vector and Tensor Mathematics

The variables used to describe physical quantities are of a number of types, including scalars, vectors, and tensors. Effort has been made in the notes to indicate these types consistently as follows:

\[ s = \text{scalar, lightface italic} \]

\[ \mathbf{v} = \text{vector, boldface with single underscore} \]

\[ \mathbf{T} = \text{tensor, boldface with double underscore}. \]  (3.10)

Scalars are used to represent physical quantities with no directional qualities, such as temperature, volume, and time. Vectors are used for quantities which have a single directional quality such as velocity and force. Tensors (we will consider only second-order tensors) are associated with quantities which have two directional characteristics, such as a momentum flux.

3.2.1 Vectors and Vector Operations

Given a coordinate system in three dimensions, a vector may thus be represented by an ordered set of three components which represent its projections \( v_1, v_2, v_3 \) on the coordinate axes 1, 2, 3:

\[ \mathbf{v} = [v_1, v_2, v_3]. \]  (3.11)

The three most commonly used coordinate systems are rectangular, cylindrical, and spherical, as described in the Coordinate Systems Notebook. Alternatively, a vector may be represented by the sum of the magnitudes of its projections on three mutually perpendicular axes:

\[ \mathbf{v} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3 = \sum_{i=1}^{3} v_i \hat{e}_i. \]  (3.12)

The unit vectors \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \) are \( \hat{e}_x, \hat{e}_y, \hat{e}_z \) in the rectangular coordinate system, \( \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi \) in cylindrical coordinates and \( \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi \) in spherical. Each of these unit vectors points in the direction of the indicated spatial coordinate, and has a magnitude of exactly one. The formulas for the vector and tensor operations described below are generally applicable to these three coordinate systems, with the exception of differential operators. Differential operators in
cylindrical and spherical coordinates must be handled more explicitly because in those cases \( \hat{\mathbf{e}}_i \) are not constant in direction (with the sole exception of \( \hat{\mathbf{e}}_z \) in cylindrical coordinates).

The magnitude of a vector is given by:

\[
|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{\sum_{i=1}^{3} v_i^2} .
\]  

(3.13)

Addition and subtraction of vectors is easily executed:

\[
\mathbf{v} + \mathbf{w} = (v_1 + w_1)\hat{\mathbf{e}}_1 + (v_2 + w_2)\hat{\mathbf{e}}_2 + (v_3 + w_3)\hat{\mathbf{e}}_3 = \sum_{i=1}^{3} (v_i + w_i)\hat{\mathbf{e}}_i ,
\]  

(3.14)

as well as multiplication by a scalar

\[
sv = (sv_1)\hat{\mathbf{e}}_1 + (sv_2)\hat{\mathbf{e}}_2 + (sv_3)\hat{\mathbf{e}}_3 = s \sum_{i=1}^{3} v_i\hat{\mathbf{e}}_i .
\]  

(3.15)

The dot product of two vectors results in a scalar:

\[
\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3 = \sum_{i=1}^{3} v_iw_i .
\]  

(3.16)

### 3.2.2 Tensors and Tensor Operations

A tensor is similarly represented by an ordered array of nine components:

\[
\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} .
\]  

(3.17)

The diagonal elements of a tensor are those which have two identical subscripts, while the other elements are termed nondiagonal. The transpose of a tensor is obtained by interchanging the subscripts on each element:

\[
\mathbf{T}^T = \begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} .
\]  

(3.18)
A tensor is described as symmetric when $T = T^T$. One special tensor is the unit tensor:

$$\delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (3.19)$$

The dyadic product of two vectors results in a tensor, as follows:

$$v \cdot w = \begin{bmatrix} v_1w_1 & v_1w_2 & v_1w_3 \\ v_2w_1 & v_2w_2 & v_2w_3 \\ v_3w_1 & v_3w_2 & v_3w_3 \end{bmatrix}. \quad (3.20)$$

This leads to the definition of the unit dyads, of which there are nine:

$$\delta_1 \cdot \delta_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (3.21)$$

$$\delta_1 \cdot \delta_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ etc.} \quad (3.22)$$

In a similar manner to vectors, tensors are easily added

$$T + U = \begin{bmatrix} T_{11} + U_{11} & T_{12} + U_{12} & T_{13} + U_{13} \\ T_{21} + U_{21} & T_{22} + U_{22} & T_{23} + U_{23} \\ T_{31} + U_{31} & T_{32} + U_{32} & T_{33} + U_{33} \end{bmatrix} = \sum_{i=1}^{3} \sum_{j=1}^{3} (T_{ij} + U_{ij}) \delta_i \delta_j, \quad (3.23)$$

or multiplied by scalars:

$$sT = \begin{bmatrix} sT_{11} & sT_{12} & sT_{13} \\ sT_{21} & sT_{22} & sT_{23} \\ sT_{31} & sT_{32} & sT_{33} \end{bmatrix} = s \sum_{i=1}^{3} \sum_{j=1}^{3} T_{ij} \delta_i \delta_j. \quad (3.24)$$

The double dot product of two tensors results in a scalar:

$$T : U = \begin{bmatrix} T_{11}U_{11} + T_{12}U_{21} + T_{13}U_{31} \\ T_{21}U_{12} + T_{22}U_{22} + T_{23}U_{32} \\ T_{31}U_{13} + T_{32}U_{23} + T_{33}U_{33} \end{bmatrix} = \sum_{i=1}^{3} \sum_{j=1}^{3} T_{ij}U_{ji}. \quad (3.25)$$

The dot product of a tensor with a vector is:

$$T \cdot v = \delta_1(T_{11}v_1 + T_{12}v_2 + T_{13}v_3) + \delta_2(T_{21}v_1 + T_{22}v_2 + T_{23}v_3) + \delta_3(T_{31}v_1 + T_{32}v_2 + T_{33}v_3) = \sum_{i=1}^{3} \delta_i \left( \sum_{j=1}^{3} T_{ij}v_j \right). \quad (3.26)$$
In contrast, the dot product of a vector with a tensor is:

\[ \mathbf{v} \cdot \mathbf{T} = \frac{\delta_1}{\delta_2}(v_1T_{11} + v_2T_{21} + v_3T_{31}) \]

\[ + \frac{\delta_2}{\delta_3}(v_1T_{12} + v_2T_{22} + v_3T_{32}) \]

\[ + \frac{\delta_3}{\delta_1}(v_1T_{13} + v_2T_{23} + v_3T_{33}) \]

\[ = \sum_{i=1}^{3} \delta_i \left( \sum_{j=1}^{3} v_j T_{ji} \right). \] (3.27)

In general, \((\mathbf{T} \cdot \mathbf{v}) \neq (\mathbf{v} \cdot \mathbf{T})\), however, they are equal if \(\mathbf{T}\) is symmetric.

The magnitude of a tensor is defined as:

\[ |\mathbf{T}| = \sqrt{\frac{1}{2}(\mathbf{T} : \mathbf{T}')} = \sqrt{\frac{1}{2} \sum_{i} \sum_{j} T_{ij}^2}. \] (3.28)

### 3.2.3 Further Reading

A simple introduction to vectors and tensors is provided by:


Numerous problems, some with solutions may be found in:


An excellent discussion of vector and tensor notation which is particularly relevant to polymer processing is in Appendix A of:


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